Notes on Inequalities

1. Properties of Absolute Values of Real Numbers
   The absolute value of a real number x is defined as

   $$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

   For any real numbers x, a, and b,the following relations hold:

   a) $|x| \geq 0$ and is equal to zero only when $x = 0$.

   b) $|-x| = |x|$.

   c) $|x|^2 = x^2$.

   d) $|ab| = |a| \cdot |b|$.

   e) $|\frac{a}{b}| = \frac{|a|}{|b|}$.

2. Triangle Inequality
   For real numbers $x_1, x_2, \ldots, x_n$,

   $$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|.$$

   The equality holds when all the x_i have the same sign.

   Another version of this inequality is

   $$|\pm x_1 \pm x_2 + \cdots + \pm x_n| \leq |x_1| + |x_2| + \cdots + |x_n|.$$

3. AM-GM Inequality
   For non-negative real numbers $x_1, x_2, \ldots, x_n$,

   $$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}$$

The equality holds when $x_1 = x_2 = \cdots = x_n$.

4. HM-GM-AM Inequality
For $x_1,\, x_2,\, \ldots,\, x_n \in \mathbb{R}^+$,

$$\frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}} \leq \sqrt[n]{x_1 x_2 \ldots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

The equality holds on both sides when $x_1 = x_2 = \cdots = x_n$.

5. Rearrangement Inequality
If $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ are two collections of real numbers in increasing order, then for any permulation $(a_1',\, a_2',\, \cdots,\, a_n')$ of $(a_1,\, a_2,\, \cdots,\, a_n)$,

$$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \geq a_1' b_1 + a_2' b_2 + a_n' b_n.$$

The equality hold if and only if $(a_1',\, a_2',\, \cdots,\, a_n') = (a_1,\, a_2,\, \cdots,\, a_n)$.
It follows that

$$a_1 b_1 + a_2 b_2 + \ldots a_n b_n \geq a_n b_1 + a_{n-1} b_2 + \cdots + a_1 b_n.$$

6. Cauchy-Schwarz Inequality
For real numbers $x_1,\, x_2,\, \ldots\,,\, x_n$,

$$\left(\sum_{i=1}^{n} x_i y_i \right)^2 \leq \left(\sum_{i=1}^{n} x_i^2 \right) \left(\sum_{i=1}^{n} y_i^2 \right).$$

The equality holds if and only if there exists some $\lambda \in \mathbb{R}$ such that $x_i = \lambda y_i$ for all $i = 1, 2, \ldots, n$.

7. Cauchy-Schwarz Inequality in Engel Form
For real numbers $a_1,\, a_2,\, \ldots\,,\, a_n$ and $x_1,\, x_2,\, \ldots\,,\, x_n > 0$,

$$\frac{a_1^2}{x_1} + \frac{a_2^2}{x_n} + \cdots + \frac{a_n^2}{x_n} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{x_1 + x_2 + \cdots + x_n}.$$

The equality holds if and only if

$$\frac{a_1}{x_1} = \frac{a_2}{x_2} = \cdots = \frac{a_n}{x_n}.$$

8. Quadratic Mean-Arithmetic Mean Inequality
For $x_1,\, x_2,\, \ldots,\, x_n \in \mathbb{R}^+$,

$$\sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}} \geq \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

9. Tchebychev's Inequality
If $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ are two collections of real numbers,

$$\frac{a_1b_1 + a_2b_2 + \ldots a_nb_n}{n} \geq \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)\left(\frac{b_1 + b_2 + \cdots + b_n}{n}\right).$$

The equality holds when $a_1 = a_2 = \cdots = a_n$ or $b_1 = b_2 = \cdots = b_n$.

10. Geometric Inequalities
a) If A, B, and C are points on a plane, then

$$AB + BC \geq AC.$$

The equality holds if and only if B lies on the line segment AC.

b) In a triangle, the longest side is the opposite to the greatest angle and vice versa.
Hence, if in a triangle ABC we have $\angle A > \angle B$, then $BC > CA$.

11. Ptolemy's Inequality
If $ABCD$ is a convex quadrilateral, then

$$AC \cdot BD \leq (AB \cdot CD + BC \cdot AD).$$

The equality holds if and only if $ABCD$ is a cyclic quadrilateral.

12. Euler's Inequality
If R and r are the circumradius and the inradius of a triangle ABC,

$$R \geq 2r.$$

The equality holds if and only if the triangle ABC is equilateral.

13. Leibniz's Inequality
In a triangle ABC with side lengths a, b, and c and circumradius R, the following relation holds:

$$9R^2 \geq a^2 + b^2 + c^2.$$

14. Symmetric Functions of the Sides of a Triangle
If a, b, and c are the lengths of the sides of a triangle, s is its semiperimeter, r is its inradius, and R is its circumradius, the following relations hold:

$$a + b + c = 2s$$

$$ab + bc + ca = s^2 + r^2 + 4rR$$

$$abc = 4Rrs$$

$$s^3 - (a + b + c)s^2 + (ab + bc + ca)s - abc = r^2 s$$

$$a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr)$$

$$a^3 + b^3 + c^3 = 2(s^3 - 3r^2 s - 6Rrs)$$