Complex Numbers Simplified

1. Complex Numbers
   Numbers of the form $x + yi$, where $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$. The set of complex numbers is denoted by $\mathbb{C}$.

2. Properties of Complex Numbers
   Commutative Law for Addition

   $$z_1 + z_2 = z_2 + z_1$$

   for all $z_1, z_2 \in \mathbb{C}$.

   Associative Law for Addition

   $$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$$

   for all $z_1, z_2, z_3 \in \mathbb{C}$.

   Additive Identity

   $$z + 0 = 0 + z = z$$

   for all $z \in \mathbb{C}$ (and $0 = 0 + i \cdot 0$).

   Additive Inverse

   For any complex number $z = x + yi$, there is a unique $-z = -x - yi \in \mathbb{C}$ such that

   $$z + (-z) = 0$$

   Commutative Law for Multiplication

   $$z_1 \cdot z_2 = z_2 \cdot z_1$$

   for all $z_1, z_2 \in \mathbb{C}$.

   Associative Law for Multiplication

   $$(z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3)$$

   for all $z_1, z_2, z_3 \in \mathbb{C}$.

   Multiplicative Identity

   $$z \cdot 1 = 1 \cdot z = z$$

for all $z \in \mathbb{C}$ (and $1 = 1 + i \cdot 0$).

Multiplicative Inverse
For any complex number $z = x + yi$, $z \neq 0$, there is a unique $z^{-1} \in \mathbb{C}$ such that

$$z \cdot z^{-1} = z^{-1} \cdot z = 1$$

3. Distributive Law

$$z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$$

for all $z_1, z_2, z_3 \in \mathbb{C}$.

4. Algebraic Properties of Complex Numbers

$$z^m \cdot z^n = z^{m+n}$$

$$\frac{z^m}{z^n} = z^{m-n}, z \neq 0$$

$$(z^m)^n = z^{mn}$$

$$(z_1 \cdot z_2)^n = z_1^n \cdot z_2^n$$

$$\left(\frac{z_1}{z_2}\right)^n = \frac{z_1^n}{z_2^n}$$

For $z = 0$, $0^n = 0$ for all integers $n > 0$.

5. Powers of Number i
$i^0 = 1; i^1 = i; i^2 = -1; i^3 = -i$
$i^{n+4} = i^n$, for $n \geq 0$

6. Conjugate of a Complex Number
For a complex number $z = x + yi$, the number $\bar{z} = x - yi$ is called the complex conjugate of z.

7. Properties of Complex Numbers in terms their Conjugates
   The relation $z = \bar{z}$ holds if and only if $z \in \mathbb{R}$ or in other words the imaginary component is zero.
   $z = \bar{\bar{z}}$.
   $z \cdot \bar{z}$ is a nonnegative real number.
   $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$.
   $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$.
   $\overline{z^{-1}} = (\bar{z})^{-1}$
   $\overline{(\frac{z_1}{z_2})} = \frac{\bar{z}_1}{\bar{z}_2}$, $z_2 \neq 0$.
   If $z = x + yi$, $x = Re(z) = \frac{z+\bar{z}}{2}$; $y = Im(z) = \frac{z-\bar{z}}{2i}$.

8. Modulus of a Complex Number
   The number $|z| = \sqrt{x^2 + y^2}$ is called the modulus or the absolute value of the complex number $z = x + yi$.

9. Polar Representation of a Complex Number
   A complex number $z = x + yi$ can be expressed as

   $$z = r(\cos\theta + i\sin\theta),$$

   where $r \in [0, \infty)$ and $\theta \in [0, 2\pi)$.
   r is called the polar radius of z and is equal to the modulus of z. The polar argument θ of z is called the argument of z.

10. Euler's Formula
    $e^{i\theta} = \cos\theta + i\sin\theta$

11. De Moivre's Theorem
    For $z = r(\cos\theta + i\sin\theta$, and $n \in \mathbb{N}$, we have

    $$z^n = r^n(\cos n\theta + i\sin n\theta).$$

12. Roots of Complex Numbers
    If n is a positive integer and $z = r(\cos\theta + i\sin\theta)$ is a complex number, then the number z has n distinct n^{th} roots given bythe formula

    $$Z_k = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right),$$

    where $k = 0, 1, 2, \ldots n - 1$.