Notes on Plane Trigonometry - Part 2

1. Sine Law
   Let ABC be a triangle with angles $\angle A$, $\angle B$ and $\angle C$, and a, b, and c as side lengths opposite to them respectively.

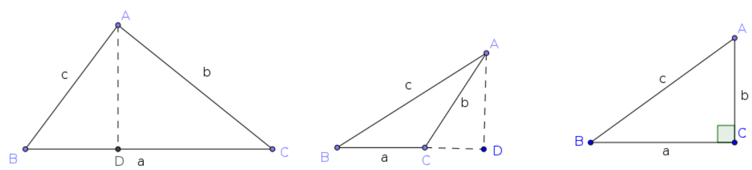

Figure 1. Sine Law

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{1}$$

Proof:
Refer to Figure 1. Draw AD perpendicular to BC (or BC extended if $\angle C$ is obtuse. AD will coincide with AC if $\angle C$ is right angle).
$AD = AB \sin B = AC \sin C$ ($AD = AB \sin B = AC \sin(180^\circ - C)$ if $\angle C$ is obtuse)
$\Rightarrow AD = c \sin B = b \sin C$
$\therefore$

$$\frac{\sin B}{b} = \frac{\sin C}{c} \tag{2}$$

Similarly, we can show that

$$\frac{\sin A}{a} = \frac{\sin B}{b} \tag{3}$$

From (2) and (3), we get

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{4}$$

We can also prove that
$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{1}{2R} = \frac{2W}{abc}$

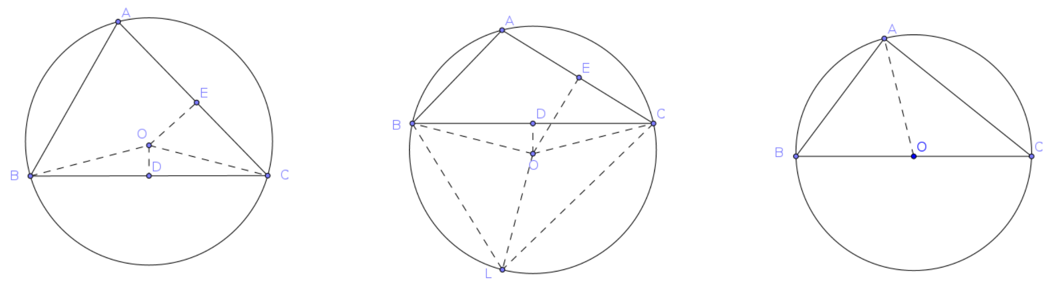

Figure 2. Relation with Circumradius

We have already proved the Sine Law relation earlier. To prove the relation with circumradius, we consider also three cases where $\angle A$ is acute, obtuse, and a right angle as shown in Figure 2.

Case 1. $\angle A$ is acute

Draw the circumcircle of $\triangle ABC$ with O as the circumecenter and R as the circumradius. From O, draw perpendicular bisectors of BC and AC meeting BC and AC at D and E respectively.
$\triangle BOD$ and $\triangle COD$ are congruent. (SSS)

$\therefore \angle BOD = \angle COD = \frac{1}{2}\angle BOC = \angle A$

$\Rightarrow BD = OB \sin \angle BOD = R \sin A$

$\Rightarrow a = BC = 2R \sin A$

or

$$\frac{\sin A}{a} = \frac{1}{2R} \tag{5}$$

From (4) and (5), we have

$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{1}{2R}$

Case 2. $\angle A$ is obtuse

Draw the circumcircle of $\triangle ABC$ with O as the circumecenter and R as the circumradius. From O, draw perpendicular bisectors of BC and AC meeting BC and AC at D and E respectively.
$\triangle BOD$ and $\triangle COD$ are congruent. (SSS)

$\therefore \angle BOD = \angle COD = \frac{1}{2}\angle BOC = \angle BLC = 180° - \angle A$

$\Rightarrow BD = OB \sin \angle BOD = R \sin(180° - A) = R \sin A$

$\Rightarrow a = BC = 2R \sin A$

or
$\frac{\sin A}{a} = \frac{1}{2R}$

and the result follows.

Case 3. $\angle A$ is a right angle

$2R = a = \frac{a}{\sin A}$, since $\angle A$ is a right angle

and the result follows.
So for all three cases, we have

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{1}{2R} \tag{6}$$

Finally, we want to show the area relationship.

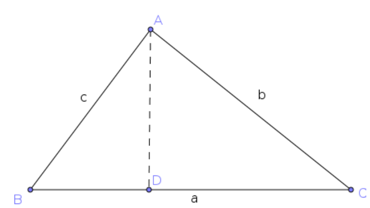

Figure 3. Area Relationship

Refer to figure 3.
Let W be the area of the triangle.
$W = \frac{1}{2}BC \cdot AD = \frac{1}{2}BC \cdot AC \sin C = \frac{1}{2}ab \sin C$
$\Rightarrow \frac{\sin C}{c} = \frac{2W}{abc}$

It can be seen that the result holds regardless of whether $\angle C$ is obtuse or a right angle.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{1}{2R} = \frac{2W}{abc} \tag{7}$$

2. Cosine Law

   Refer to Figure 1.
   Consider the case where $\angle C$ is acute.

$$\begin{aligned}
AB^2 &= AD^2 + BD^2 \\
&= AD^2 + (BC - CD)^2 \\
&= AD^2 + BC^2 + CD^2 - 2BC \cdot CD \\
&= AD^2 + BC^2 + CD^2 - 2BC \cdot AC \cos C \\
&= (AD^2 + CD^2) + BC^2 - 2BC \cdot AC \cos C \\
&= AC^2 + BC^2 - 2BC \cdot AC \cos C \\
\therefore c^2 &= b^2 + a^2 - 2ab \cos C \\
or &
\end{aligned}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \tag{8}$$

   Following similar arguments, we can prove the result for the cases where C is an obtuse angle or a right angle.
   Similarly, it can be proved that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \tag{9}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca} \tag{10}$$

3. Radius of the Incircle
   Refer to Figure 4.

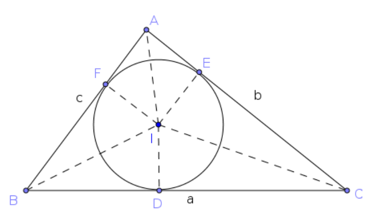

Figure 4. Incircle of Triangle ABC

   Let I be the center of the incircle with radius r. Let AI, BI, and CI be the bisectors of angles A, B, and C respectively.
   Drop ID, IE, and IF perpendicular to sides BC, AC, and AB respectively.

Let W be the area of $\triangle ABC$ and $s = \frac{a+b+c}{2}$ its semiperimeter.
$ID = IE = IF = r$
Area of $\triangle ABC =$ Area of $\triangle IBC +$ area of $\triangle IAC +$ area of $\triangle IAB$
or
$W = \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr = r \cdot \frac{a+b+c}{2} = rs$

$\therefore r = \frac{W}{s}$

Triangles IBD and IBF are congruent.

$\therefore BD = BF$

Similarly, $AE = AF$ and $CE = CD$.

$\therefore 2BD + 2AE + 2CE = (BD + BF) + (AE + AF) + (CE + CD)$

$\therefore 2BD + 2AC = BC + AC + AB$
or
$2BD + 2b = 2s$
or
$BD = s - b$

Similarly, $CE = s - c$ and $AF = s - b$

$\frac{ID}{BD} = \tan \frac{B}{2}$

$\Rightarrow r = BD \tan \frac{B}{2} = (s - b) \tan \frac{B}{2}$

Similarly,
$r = AF \tan \frac{A}{2} = (s - a) \tan \frac{A}{2}$

$r = CE \tan \frac{C}{2} = (s - c) \tan \frac{C}{2}$