Notes on Plane Trigonometry - Part 1

1. Circular Functions

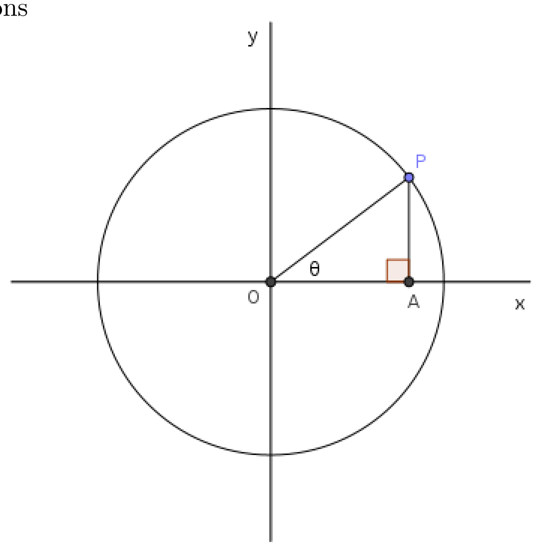

Figure 1. Circular Functions

Consider a circle (radius = OP) in the XY plane as shown in Figure 1. Let OP make angle θ with the X-axis and let PA be the perpendicular projection of P to the X-axis.

<u>Definitions</u>

$$\sin\theta = \frac{PA}{OP} \tag{1}$$

$$\cos\theta = \frac{OA}{OP} \tag{2}$$

$$\tan\theta = \frac{PA}{OA} \tag{3}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{OA}{PA} \tag{4}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{OP}{PA} \tag{5}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{OP}{OA} \tag{6}$$

θ is a real number. If point P lies in the first quadrant depending on the value of θ, distances PA and OA are both positive. If it lies in the second quadrant, distance PA is positive whereas distance OA is negative. If P lies in the third quadrant, both PA and OA are negative. If P lies in the fourth quadrant, PA is negative while OA is positive. In evaluating the circular functions, distance OP is the magnitude of the radius. Consequently, trigonometric functions can have positive, negative, or zero values.

Functions $\tan\theta$ and $\sec\theta$ are undefined for $\theta = n\pi + \frac{\pi}{2}$ where n is an integer.
Similarly, $\cot\theta$ and $\csc\theta$ are undefined for $\theta = n\pi$ where n is an integer.

Figure 2 shows the polarities of trigonometric functions for angle θ lying in the four quadrants.

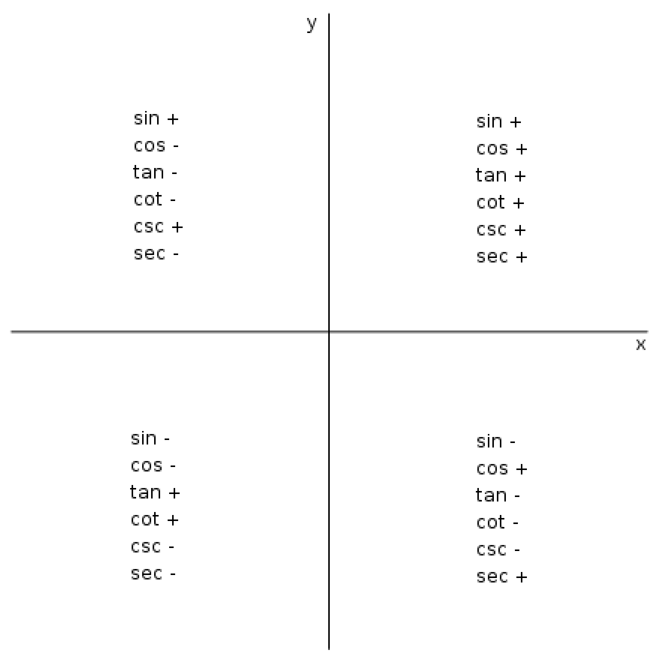

Figure 2. Polarity of Trigonometric Functions in Four Quadrants

2. Basic Relations between Trigonometrical Ratios

$$\sin^2\theta + \cos^2\theta = 1 \tag{7}$$

$$\tan^2\theta + 1 = \sec^2\theta \tag{8}$$

$$\cot^2\theta + 1 = \csc^2\theta \tag{9}$$

The above equations can be easily derived by applying Pythagoras' theorem to $\triangle OPA$ in Figure 1 and using definitions in equations 1 through 6.

3. Trigonometric Functions in Terms of One Another

Function	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
$\sin\theta$	$\sin\theta$	$\sqrt{1-\cos^2\theta}$	$\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$	$\frac{1}{\sqrt{1+\cot^2\theta}}$	$\frac{\sqrt{\sec^2\theta-1}}{\sec\theta}$	$\frac{1}{\csc\theta}$
$\cos\theta$	$\sqrt{1-\sin^2\theta}$	$\cos\theta$	$\frac{1}{\sqrt{1+\tan^2\theta}}$	$\frac{\cot\theta}{\sqrt{1+\cot^2\theta}}$	$\frac{1}{\sec\theta}$	$\frac{\sqrt{\csc^2\theta-1}}{\csc\theta}$
$\tan\theta$	$\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$	$\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$	$\tan\theta$	$\frac{1}{\cot\theta}$	$\sqrt{\sec^2\theta-1}$	$\frac{1}{\sqrt{\csc^2\theta-1}}$
$\cot\theta$	$\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$	$\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}$	$\frac{1}{\tan\theta}$	$\cot\theta$	$\frac{1}{\sqrt{\sec^2\theta-1}}$	$\sqrt{\csc^2\theta-1}$
$\sec\theta$	$\frac{1}{\sqrt{1-\sin^2\theta}}$	$\frac{1}{\cos\theta}$	$\sqrt{1+\tan^2\theta}$	$\frac{\sqrt{1+\cot^2\theta}}{\cot\theta}$	$\sec\theta$	$\frac{\csc\theta}{\sqrt{\csc^2\theta-1}}$
$\csc\theta$	$\frac{1}{\sin\theta}$	$\frac{1}{\sqrt{1-\cos^2\theta}}$	$\frac{\sqrt{1+\tan^2\theta}}{\tan\theta}$	$\sqrt{1+\cot^2\theta}$	$\frac{\sec\theta}{\sqrt{\sec^2\theta-1}}$	$\csc\theta$

4. Trigonometric Ratios for Special Angles

Angle	$0°$	$30°$	$45°$	$60°$	$90°$
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Tangent	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undef
Cotangent	undef	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0
Cosecant	undef	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1
Secant	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	undef

5. Graphs of Trigonometric Functions

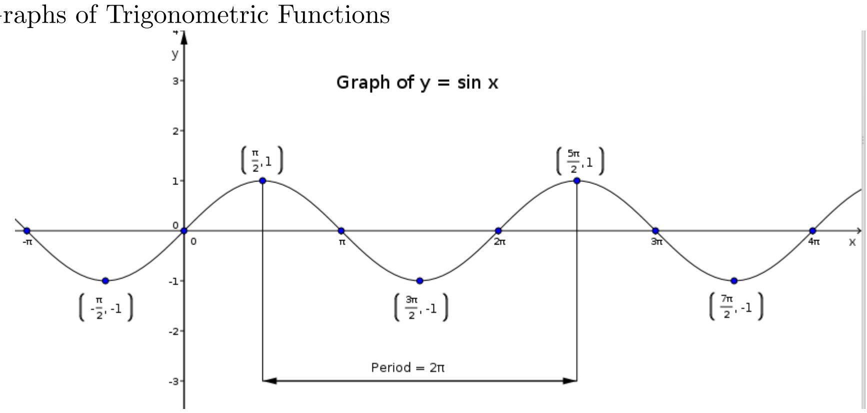

Figure 3. Graph of Sine Function

Domain: $-\infty < x < \infty$
Range: $-1 \leq y \leq 1$
Period: 2π

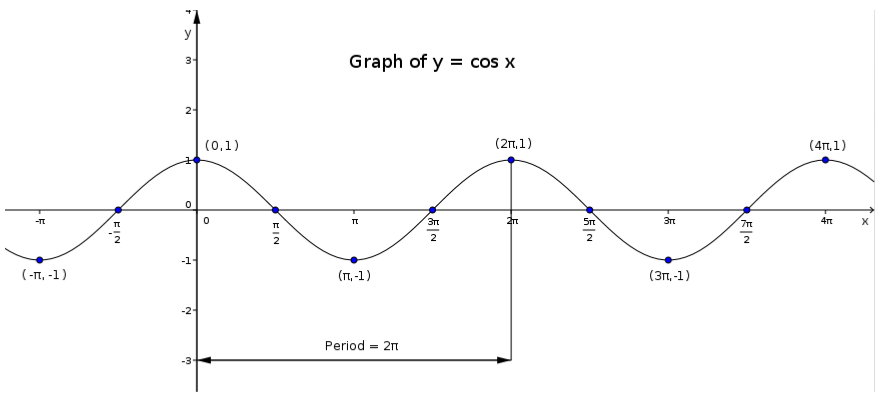

Figure 4. Graph of Cosine Function

Domain: $-\infty < x < \infty$
Range: $-1 \leq y \leq 1$
Period: 2π

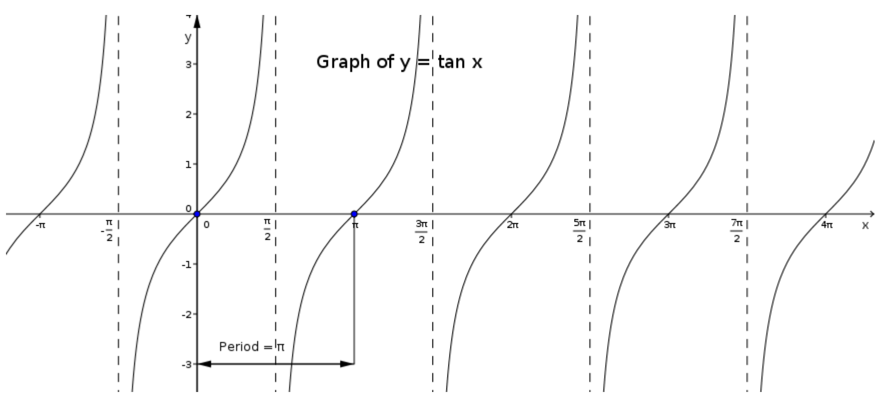

Figure 5. Graph of Tangent Function

Domain: $\frac{(2n-1)\pi}{2} < x < \frac{(2n+1)\pi}{2}$, where n is an integer.
Range: $-\infty < y < \infty$
Period: π

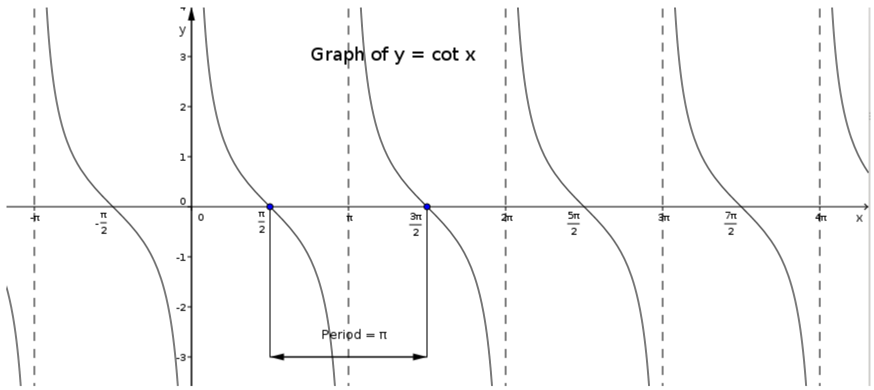

Figure 6. Graph of Cotangent Function

Domain: $n\pi < x < (n+1)\pi$, where n is an integer.
Range: $-\infty < y < \infty$
Period: π

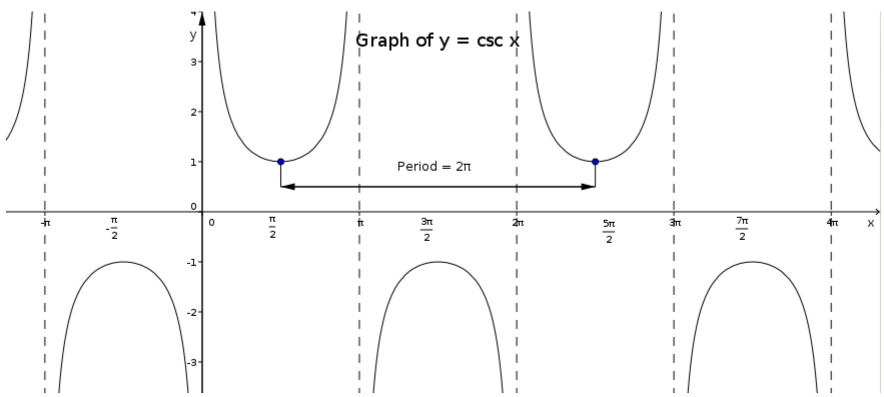

Figure 7. Graph of Cosecant Function

Domain: $n\pi < x < (n+1)\pi$, where n is an integer.
Range: $-\infty < y \leq 1$ and $1 \leq y < \infty$
Period: 2π

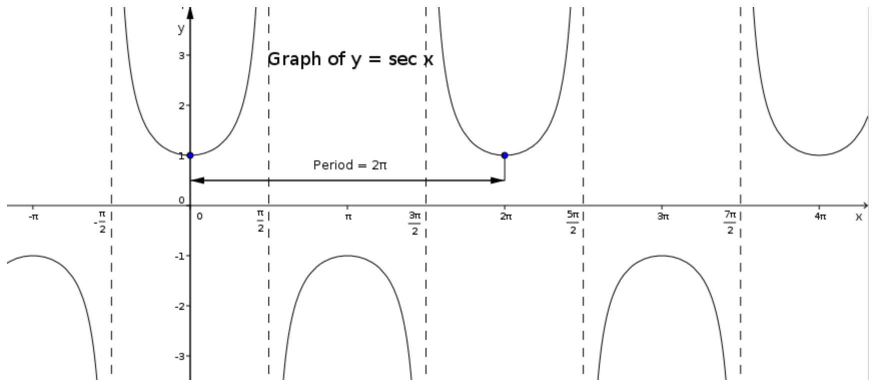

Figure 8. Graph of Secant Function

Domain: $\frac{(2n-1)\pi}{2} < x < \frac{(2n+1)\pi}{2}$, where n is an integer.
Range: $-\infty < y < \infty$
Period: 2π

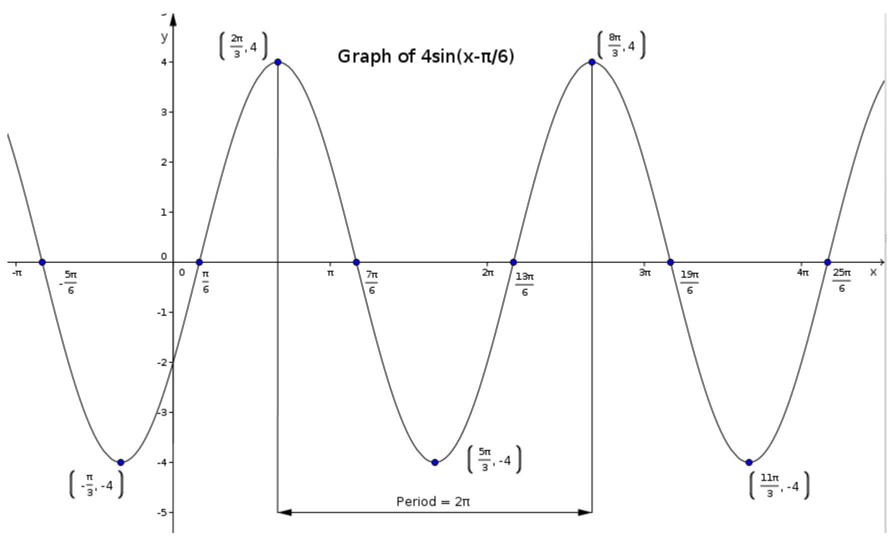

Figure 9. Graph of Function $y = 4\sin(x - \frac{\pi}{6})$

Domain: $-\infty < x < \infty$
Range: $-4 \leq y \leq 4$
Period: 2π
Phase Shift: $\frac{\pi}{6}$

6. Miscellaneous Trigonometric Relations

$$\sin(-\theta) = -\sin\theta \tag{10}$$

$$\cos(-\theta) = \cos\theta \tag{11}$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \tag{12}$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta \tag{13}$$

$$\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta \tag{14}$$

$$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta \tag{15}$$

$$\sin(\pi - \theta) = \sin\theta \tag{16}$$

$$\cos(\pi - \theta) = -\cos\theta \tag{17}$$

$$\sin(\pi + \theta) = -\sin\theta \tag{18}$$

$$\cos(\pi + \theta) = -\cos\theta \tag{19}$$

$$\sin(2\pi + \theta) = \sin\theta \tag{20}$$

$$\cos(2\pi + \theta) = \cos\theta \tag{21}$$

7. General Expression for Angles Having the Same Trigonometric Ratio
If $\sin\theta = \sin\alpha$, then

$$\theta = n\pi + (-1)^n\alpha \tag{22}$$

where n is an integer.
If $\cos\theta = \cos\alpha$, then

$$\theta = 2n\pi \pm \alpha \tag{23}$$

where n is an integer.
If $\tan\theta = \tan\alpha$, then

$$\theta = n\pi + \alpha \tag{24}$$

where n is an integer.

8. Trigonometric Ratios for the Sum and the Difference of Two Angles

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi \tag{25}$$

$$\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi \tag{26}$$

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi \tag{27}$$

$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi \tag{28}$$

$$\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} \tag{29}$$

$$\tan(\theta - \phi) = \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi} \tag{30}$$

9. Product Formulae

$$\sin\theta + \sin\phi = 2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right) \tag{31}$$

$$\sin\theta - \sin\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right) \tag{32}$$

$$\cos\theta + \cos\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right) \tag{33}$$

$$\cos\theta - \cos\phi = -2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right) \tag{34}$$

$$2\sin\theta\cos\phi = \sin(\theta+\phi) + \sin(\theta-\phi) \tag{35}$$

$$2\cos\theta\sin\phi = \sin(\theta+\phi) - \sin(\theta-\phi) \tag{36}$$

$$2\cos\theta\cos\phi = \cos(\theta+\phi) + \cos(\theta-\phi) \tag{37}$$

$$2\sin\theta\sin\phi = \cos(\theta-\phi) + \cos(\theta+\phi) \tag{38}$$

10. Multiple Angle Formulae

$$\sin 2\theta = 2\sin\theta\cos\theta \tag{39}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta \tag{40}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} \tag{41}$$

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta \tag{42}$$

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta \tag{43}$$

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta} \tag{44}$$

11. Half Angle Formulae

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}} \tag{45}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}} \tag{46}$$

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \tag{47}$$

12. Trigonometric Ratios in Triangles

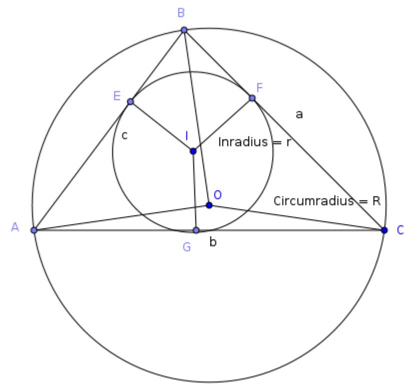

Figure 10. A Triangle with a Circumcircle and an Incircle

Let ABC be a triangle with angles $\angle A$, $\angle B$ and $\angle C$, and a, b, and c as side lengths opposite to them respectively.
Furthermore, let R be the circumradius of the triangle and $s = \frac{a+b+c}{2}$ be the semiperimeter of the triangle. Also, let W be the area of the triangle.
Area of $\triangle ABC$

$$W = \sqrt{s(s-a)(s-b)(s-c)} = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C = \frac{abc}{4R} \tag{48}$$

Sine Law

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{1}{2R} = \frac{2W}{abc} \tag{49}$$

Cosine Law

$$\cos A = \frac{b^2+c^2-a^2}{2bc} \tag{50}$$

$$\cos B = \frac{c^2+a^2-b^2}{2ca} \tag{51}$$

$$\cos C = \frac{a^2+b^2-c^2}{2ab} \tag{52}$$

Other Relations

$$r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2} = \frac{W}{s} \tag{53}$$

$$r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \tag{54}$$

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \tag{55}$$

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} \tag{56}$$

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \tag{57}$$

Similar relations can be written for trigonometric ratios of $\frac{B}{2}$ and $\frac{C}{2}$.

Distance between Circumcenter and Incenter

$$OI = R\sqrt{1 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} \tag{58}$$

$$OI^2 = R^2 - 2Rr \tag{59}$$

13. Inverse Circular Functions

Note that the trigonometric function are periodic functions and have many-to-one mapping going from domain to range. Furthermore, functions such as $\sin\theta$, $\cos\theta$, $\csc\theta$, and $\sec\theta$ have a restricted range in the space of real numbers. Strictly speaking, inverse functions for these functions cannot exist. However, inverse functions have been defined with restrictions on the domain of such functions so that a minimal value for an angle is obtained.

$\sin^{-1} a = \theta$ where $-1 \le a \le 1$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\cos^{-1} a = \theta$ where $-1 \le a \le 1$ and $0 \le \theta \le \pi$

$\tan^{-1} a = \theta$ where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\cot^{-1} a = \theta$ where $0 < \theta < \pi$

$\csc^{-1} a = \theta$ where $a \le -1$ and $a \ge 1$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\sec^{-1} a = \theta$ where $a \le -1$ and $a \ge 1$ and $0 \le \theta \le \pi$